FIXED POINT THEOREM FOR NON-SELF MAPS OF REGIONS IN THE PLANE

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Abstract. Let $X \subset \mathbb{R}^2$ be a compact, simply connected and locally connected set, and let $f: X \to Y \subset \mathbb{R}^2$ be a homeomorphism isotopic to the identity on X. Generalizing Brouwer's plane translation theorem for self-maps of the plane, we prove that f has no recurrent (in particular no periodic) points if it has no fixed points.

1. Introduction and Statement of the Result

Since Brouwer's proof of his plane translation theorem [4], many alternative proofs of the theorem and its key ingredient, the translation arc lemma, have been given (for several more recent ones, see Brown [5], Fathi [7], Franks [8]). The following is a concise formulation of its main statement.

Theorem A (Barge and Franks [1])**.** *Suppose $f: \mathbb{R}^2 \to \mathbb{R}^2$ is an orientation-preserving homeomorphism of the plane. If f has got a periodic point then it has got a fixed point.*

Slightly stronger versions assume a weaker form of recurrence, for example the existence of periodic disk chains (Barge and Franks [1]), to obtain the existence of fixed points.

In this paper we consider homeomorphisms between compact sets in the plane and obtain a statement analogous to Theorem A in which we only make certain topological assumptions on the domain of the map. Our main theorem is the following:

Theorem 1.1. *Let $X \subset \mathbb{R}^2$ be a compact, simply connected, locally connected subset of the real plane and let $f: X \to Y \subset \mathbb{R}^2$ be a homeomorphism isotopic to the identity on X. Let C be a connected component of $X \cap Y$. If f has got a periodic orbit in C, then f also has got a fixed point in C.*

In fact, as a corollary we will obtain a slightly stronger result:

Corollary 1.2. *Let $X \subset \mathbb{R}^2$ and $f: X \to Y$ be as in Theorem 1.1 and let C be a connected component of $X \cap Y$. If f has got no fixed point in C, then the orbit of every point $x \in C$ eventually leaves C, i.e. there exists $n = n(x) \in \mathbb{N}$, such that $f^n(x) \notin C$.*

In particular, if $X \cap Y$ is connected and f has got no fixed points, then the non-escaping set of f is empty:

$$\{x \in X : f^n(x) \in D \ \forall n \in \mathbb{N}\} = \emptyset.$$

To prove these results, in Section 2 we will first consider the case of an orientation-preserving homeomorphism $f: D \to E \subset \mathbb{R}^2$ of a Jordan domain D into the plane. In this special case, the statement will be first proved for $D \cap E$ connected, by suitably extending f to the real plane and applying Theorem A. We will then proceed to show that the connectedness assumption can be lifted if one formulates the result more precisely, taking into account the connected components of $D \cap E$ individually. Finally, in Section 3, we will deduce the general case of Theorem 1.1 by reducing the problem to the Jordan domain case.

In Section 4 we will discuss the assumptions of our results and questions about possible extensions.

2. Non-self maps of Jordan domains

A set $D \subset \mathbb{R}^2$ is a *Jordan domain*, if it is a closed set with boundary ∂D a simple closed curve (*Jordan curve*). By Schoenflies' theorem, Jordan domains are precisely the planar regions homeomorphic to the (closed) disk. In this paper we assume that all Jordan curves are endowed with the counter-clockwise orientation. For a Jordan curve C and $x, y \in C$, we denote by $(x,y)_C$ (resp. $[x,y]_C$) the open (resp. closed) arc in C from x to y according to this orientation.

For $X \subset \mathbb{R}^2$, we say that $x \in X$ is a *fixed point* for the map $f\colon X \to \mathbb{R}^2$, if $f(x) = x$. We say that $x \in X$ is a *periodic point* for f, if there exists $n \in \mathbb{N}$, such that $f^n(x) = x$. This of course requires that the entire *orbit* of x, $\mathcal{O}_f(x) = \{x = f^n(x), f(x), \ldots, f^{n-1}(x)\}$, is included in X.

In analogy to Theorem A, we will prove the following:

Theorem 2.1. *Let $D \subset \mathbb{R}^2$ be a Jordan domain and let $f\colon D \to E \subset \mathbb{R}^2$ be an orientation-preserving homeomorphism. Assume that $D \cap E$ is connected. If f has got a periodic point, then f also has got a fixed point.*

Remark 2.2. *All periodic or fixed points of $f\colon D \to E$ necessarily lie in $D \cap E$.*

The strategy of our proof is to show that a homeomorphism $f\colon D \to E$ as in Theorem 2.1 and without fixed points can be extended to an orientation-presering fixed point free homeomorphism $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ (a *Brouwer homeomorphism*). The result then follows from classical results such as Theorem A.

Our first step is to consider the structure of the set $D \cap E$. It is known that each connected component of the intersection of two Jordan domains is again a Jordan domain (see e.g. Kerékjártó [10, p.87]). We need the following more detailed statement (Bonino [3, Proposition 3.1], proved in Le Calvez and Yoccoz [12, Part 1]).

Proposition 2.3. *Let U, U' be two Jordan domains containing a point $o \in U \cap U'$, such that $U \not\subset U'$ and $U' \not\subset U$. Denote the connected component of $U \cap U'$ containing o by $U \wedge U'$.*

(1) *There is a partition*
$$\partial(U \wedge U') = (\partial(U \wedge U') \cap \partial U \cap \partial U') \cup \bigcup_{i\in I} \alpha_i \cup \bigcup_{j\in J} \beta_j \text{ , where}$$
- *I, J are non-empty, at most countable sets,*
- *for every $i \in I$, $\alpha_i = (a_i, b_i)_{\partial U}$ is a connected component of $\partial U \cap U'$,*
- *for every $j \in J$, $\beta_j = (c_j, d_j)_{\partial U'}$ is a connected component of $\partial U' \cap U$.*

(2) *For every $j \in J$, $U \wedge U'$ is contained in the Jordan domain bounded by $\beta_j \cup [d_j, c_j]_{\partial U}$.*

(3) *$\partial(U \wedge U')$ is homeomorphic to ∂U, hence itself a Jordan curve.*

(4) *Three points $a, b, c \in \partial(U \wedge U') \cap \partial U$ (resp. $\partial(U \wedge U') \cap \partial U'$) are met in this order on ∂U (resp. $\partial U'$) if and only if they are met in the same order on $\partial(U \wedge U')$.*

We consider a Jordan domain $D \subset \mathbb{R}^2$ and a homeomorphism $f\colon D \to E = f(D) \subset \mathbb{R}^2$. If $D \cap E$ is connected, it follows that both $D \cap E$ and $D \cup E$ are Jordan domains (see Fig.1). With Proposition 2.3 in mind, the proof of the following lemma is straightforward; it can be found in the appendix.

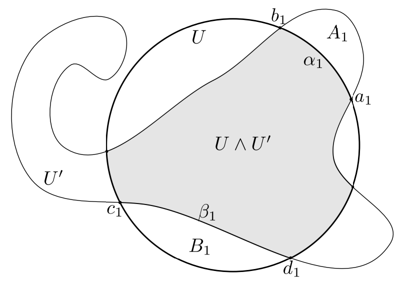


FIGURE 1. Proposition 2.3: partition of $\partial(U \wedge U')$ in the case when $U \cap U'$ is connected

Lemma 2.4. *Let $D, E \subset \mathbb{R}^2$ be Jordan domains, such that $D \not\subset E$, $E \not\subset D$ and $D \cap E$ is non-empty and connected. Then there exists a partition of $\mathbb{R}^2 \setminus \operatorname{int}(D \cap E)$ into arcs, each of which connects a point on $\partial(D \cap E)$ to ∞ and intersects each of ∂D and ∂E in precisely one point.*

The partition of $\mathbb{R}^2 \setminus \operatorname{int}(D \cap E)$ obtained this way allows us to prove our key proposition, from which the main result of this section will follow as a corollary.

Proposition 2.5. *Let $D \subset \mathbb{R}^2$ be a Jordan domain and $f : D \to E \subset \mathbb{R}^2$ an orientation-preserving homeomorphism with no fixed points, and such that $D \cap E$ is non-empty and connected. Then there exists a fixed point free orientation-preserving homeomorphism $F : \mathbb{R}^2 \to \mathbb{R}^2$ extending f (i.e., $F|_D \equiv f$).*

Proof. Since f has got no fixed points, by Brouwer's Fixed Point Theorem we get that $E \not\subset D$ and $D \not\subset E$. Therefore we can apply Lemma 2.4 to obtain a partition P of $\mathbb{R}^2 \setminus \operatorname{int}(D \cap E)$ into arcs, each connecting a point in $\partial(D \cap E)$ with ∞ and intersecting each of ∂D and ∂E in exactly one point.

For $x \in \mathbb{R}^2 \setminus \operatorname{int}(D \cap E)$, we denote by $L^x \in P$ the partition element containing the point x and let $\pi^D(x) = L^x \cap \partial D$, $\pi^E(x) = L^x \cap \partial E$. Denote by $l^D(x)$ and $l^E(x)$ the lengths of the arcs $[\pi^D(x), x]$ and $[\pi^E(x), x]$ as subarcs of L^x, respectively.

We now construct $F : \mathbb{R}^2 \to \mathbb{R}^2$ as an extension of f in such a way, that each arc $L^x \setminus D$, $x \in \partial D$, is mapped into the arc $L^{f(x)}$. For $x \in D$ we define $F(x) = f(x)$ and for $x \notin D$ we define $F(x)$ to be the unique point $y \in L^{f(\pi^D(x))} \setminus E$ such that $l^E(y) = l^D(x)$ (see Fig. 2).

The map F is an orientation-preserving homeomorphism of $\mathbb{R}^2$; its inverse can be obtained the same way by swapping the roles of D, f and E, f^{-1}. Moreover, F has got no fixed points in D since $F|_D \equiv f$. On the other hand, suppose $p \in \mathbb{R}^2 \setminus D$ with $F(p) = p$. Then $F(L^p \setminus D) \subset L^p$. Since $l^D(p) = l^E(F(p)) = l^E(p)$, we get $\pi^D(p) = \pi^E(p)$, and therefore $f(\pi^D(p)) = \pi^D(p)$, a contradiction. Hence F is fixed point free. □

Combining Proposition 2.5 and Theorem A, we now immediately obtain Theorem 2.1.

Next, we formulate a somewhat more general form of Theorem 2.1, by lifting the restriction of $D \cap f(D)$ to be connected, and instead making an assertion for each individual connected component of this set. This version of the result turns out to be much more useful in its application to more general compact sets in Section 3.

Theorem 2.6. *Let $D \subset \mathbb{R}^2$ be a Jordan domain and $f : D \to E \subset \mathbb{R}^2$ an orientation-preserving homeomorphism. Let C be a connected component of $D \cap E$. If f has got a periodic orbit in C, then f also has got a fixed point in C.*

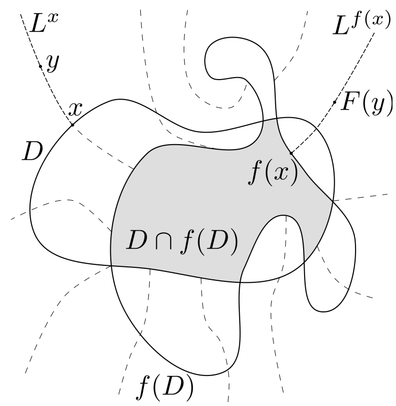


FIGURE 2. Proposition 2.5: Extension $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ of $f\colon D \to f(D)$ maps each arc L^x, $x \in \partial D$, into the arc $L^{f(x)}$.

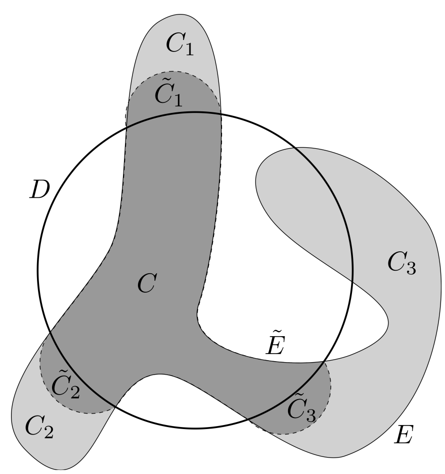


FIGURE 3. Theorem 2.6: The Jordan domains D (white), $E = (\bigcup_i C_i) \cup C$ (light grey) and $\tilde{E} = \left(\bigcup_i \tilde{C}_i\right) \cup C$ (dark grey). The $\tilde{C}_i$ are pairwise disjoint, the homeomorphism $g\colon E \to \tilde{E}$ maps each C_i homeomorphically to $\tilde{C}_i$ and fixes C pointwise. Moreover, $\tilde{E} \cap D = C$ is connected.

Proof. We show that this more general setting can be reduced to the one in Theorem 2.1. Let $\{C_i\}$ be the collection of connected components of $\overline{E \setminus C}$ and let $\gamma_i = C \cap C_i$, which is a closed subarc of ∂D (a connected component of $C \cap \partial D$).

Each C_i is a Jordan domain bounded by the union of γ_i and a closed subarc of ∂E. Let $\{\tilde{C}_i\}$ be another collection of disks $\tilde{C}_i \subset \overline{\mathbb{R}^2 \setminus D}$, such that $\tilde{C}_i \cap D = \gamma_i$ and the $\tilde{C}_i$ are pairwise disjoint. (Note that by Schoenflies' theorem we can assume without loss of generality D to be the closed standard unit disk $\overline{\mathbb{D}} = \{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 < 1\}$. Since the γ_i are pairwise disjoint, letting $\tilde{C}_i$ be the round halfdisk over $\partial\mathbb{D}$ centred at the midpoint of γ_i gives a collection of pairwise disjoint disks as required, see Fig.3.)

By Schoenflies' theorem, for each i there exists a homeomorphism $g_i \colon C_i \to \tilde{C}_i$ and we can assume that g_i is the identity on γ_i. Noting that $E = (\bigcup_i C_i) \cup C$, let $\tilde{E} = \left(\bigcup_i \tilde{C}_i\right) \cup C$

and define a homeomorphism $g\colon E \to \tilde{E}$ by gluing together the identity on C and the homeomorphisms g_i:

$$g(y) = \begin{cases} y & \text{if } y \in C, \\ g_i(y) & \text{if } y \in C_i. \end{cases}$$

Finally, define the homeomorphism $\tilde{f} = g \circ f\colon D \to \tilde{E}$. By construction, $\tilde{f}$ and f coincide on $f^{-1}(C)$, and if f has a periodic orbit in C, so does $\tilde{f}$. Since $\tilde{f}(D) \cap D = \tilde{E} \cap D = C$ is connected, we can apply Theorem 2.1 to get that $\tilde{f}$ has a fixed point in C. Hence f has a fixed point in C. □

A slightly stronger result can be formulated as a corollary to the proof of Theorem 2.6:

Corollary 2.7. *Let $f\colon D \to E$ be as in Theorem 2.6 and C a connected component of $D \cap E$. If f has got no fixed point in C, then the non-escaping set of f in C is empty:*

$$\{x \in C\colon\ f^n(x) \in C\ \forall n \in \mathbb{N}\} = \emptyset.$$

Proof. By repeating verbatim the argument in the proof of Theorem 2.6, we can construct a homeomorphism $\tilde{f}\colon D \to \tilde{E}$, which coincides with f on C and such that $\tilde{E} \cap D = C$.

Observe that $\tilde{f}$ is fixed point free, so we can apply Proposition 2.5 to get an extension of $\tilde{f}$ to a fixed point free orientation-preserving homeomorphism $F\colon \mathbb{R}^2 \to \mathbb{R}^2$. By [8, Corollary 1.3], F is a so-called *free* homeomorphism, i.e., for any topological disk $U \subset \mathbb{R}^2$ we have

$$F(U) \cap U = \emptyset \Rightarrow F^i(U) \cap F^j(U) = \emptyset \ \text{ whenever } \ i \neq j. \tag{2.1}$$

Suppose f has a point $x \in C$ with $f^n(x) \in C$ for all $n \in \mathbb{N}$. Then $f^n(x) = F^n(x)$ for all $n \in \mathbb{N}$ and the forward orbit of x has an accumulation point, $F^{n_k}(x) \to x_0 \in C$ as $k \to \infty$. A sufficiently small neighbourhood U of x_0 then satisfies $F(U) \cap U = \emptyset$ but clearly $F^{n_{k+1}-n_k}(U) \cap U \neq \emptyset$ for sufficiently large k, contradicting (2.1). Hence such point never escaping C does not exist. □

3. Non-self Maps for Compact Simply Connected Planar Sets

In this section we will generalize the results of the previous section to non-self maps of compact, simply connected and locally connected sets in the real plane (also known as *nonseparating Peano continua*). Note that in our terminology simply connected always implies connected.

We denote the Riemann sphere by $\hat{\mathbb{C}}$. If $X \subset \hat{\mathbb{C}}$ is compact, connected and nonseparating, then its complement $U = \hat{\mathbb{C}} \setminus X$ is simply connected (an open set $U \subset \hat{\mathbb{C}}$ is simply connected if and only if both U and $\hat{\mathbb{C}} \setminus U$ are connected).

The classical Riemann mapping theorem states that if $U \subsetneq \mathbb{C}$ is non-empty, simply connected and open, then there exists a biholomorphic map from U onto the open unit disk $\mathbb{D} = \{z \in \mathbb{C}\colon \|z\| < 1\}$, known as the *Riemann map*. We will make use of the following stronger result by Carathéodory (see [13, Theorem 17.14]).

Theorem B (Carathéodory's Theorem)**.** *If $U \subsetneq \hat{\mathbb{C}}$ is non-empty, simply connected and open, $\hat{\mathbb{C}} \setminus U$ has at least two points, and additionally ∂U (or $\hat{\mathbb{C}} \setminus U$) is locally connected, then the inverse of the Riemann map, $\phi\colon \mathbb{D} \to U$, extends continuously to a map from the closed unit disk $\overline{\mathbb{D}}$ onto $\overline{U}$.*

As in the case of Jordan domains, we will need a requirement on the homeomorphisms to be orientation-preserving. To make sense of this notion for more general subsets of the plane, we cite the following result by Oversteegen and Valkenburg [15], simplified by

our assumption of a nonseparating and locally connected set (see also Oversteegen and Tymchatyn [14]).

Theorem C. *Let $X \subset \hat{\mathbb{C}}$ be compact, simply connected and locally connected, and $f\colon X \to Y \subset \hat{\mathbb{C}}$ a homeomorphism. Then the following are equivalent:*

(1) *f is isotopic to the identity on X;*
(2) *there exists an isotopy $F\colon \hat{\mathbb{C}} \times [0,1] \to \hat{\mathbb{C}}$ such that $F^0 = \mathrm{id}_{\hat{\mathbb{C}}}$ and $F^1|_X = f$;*
(3) *f extends to an orientation-preserving homeomorphism of $\hat{\mathbb{C}}$;*
(4) *if $U = \hat{\mathbb{C}} \setminus X$ and $V = \hat{\mathbb{C}} \setminus Y$, then f induces a homeomorphism $\hat{f}$ from the prime end circle of U to the prime end circle of V which preserves the circular order.*

To explain assertion (4) in the above theorem, let $\phi\colon \overline{\mathbb{D}} \to \overline{U}$ and $\psi\colon \overline{\mathbb{D}} \to \overline{V}$ be the extended inverse Riemann maps given by Carathéodory's theorem, and denote $S^1 = \partial\mathbb{D}$. Then the homeomorphism $\hat{f}\colon S^1 \to S^1$ is said to be *induced by* f, if

$$\psi|_{S^1} \circ \hat{f} = f|_{\partial U} \circ \phi|_{S^1}.$$

More details can be found in [15], for an introduction to Carathéodory's theory of prime ends the reader is referred to Milnor's book [13, Chapter 17].

Furthermore, we will make use of the following notation: For $\varepsilon > 0$, denote the closed ε-neighbourhood of a set $X \subset \mathbb{R}^2$ by

$$X_\varepsilon = \{z \in \mathbb{R}^2 \colon \inf_{x \in X} \|x - z\| \le \varepsilon\}.$$

In the proof of our main theorem, we will consider ε-neighbourhoods of disconnected compact sets. One key fact we will need is that any two given connected components of such a set are separated by the ε-neighbourhood of the set, if $\varepsilon > 0$ is chosen sufficiently small. This is the following technical lemma, whose proof can be found in the appendix.

Lemma 3.1. *Let X be a compact subset of $\mathbb{R}^n$, $n \in \mathbb{N}$. Let $\mathcal{C} = \{C_i\}_{i\in I}$ be the collection of connected components of X and assume $|\mathcal{C}| \ge 2$. Let $C, C' \in \mathcal{C}$ be two distinct connected components. Then for $\varepsilon > 0$ sufficiently small, C and C' lie in two distinct connected components of X_ε.*

We are now ready to prove our main results.

Proof of Theorem 1.1. Let C be a connected component of $X \cap Y$ which contains a periodic orbit of f. We will prove that f has got a fixed point in C. The strategy of the proof is to construct Jordan-domain neighbourhoods of X and Y and to extend f to a homeomorphism between these neighbourhoods, so that Theorem 2.6 can be applied to the extension. The existence of a fixed point for the extension will then be shown to imply the existence of a fixed point for f in C.

For fixed $\varepsilon > 0$, we first construct a closed neighbourhood N^ε of X with boundary $\Gamma = \partial N^\varepsilon$ a Jordan curve, such that $\overline{N^\varepsilon \setminus X}$ has a foliation $\{\gamma_z\}_{z\in\Gamma}$ with the following properties:

- each γ_z is an arc connecting the point $z \in \Gamma$ with a point $x(z) \in \partial X$;
- $\gamma_z \cap \partial X = \{x(z)\}$ and $\gamma_z \cap \Gamma = \{z\}$ for each $z \in \Gamma$;
- $\bigcup_{z\in\Gamma} \gamma_z = \overline{N^\varepsilon \setminus X}$;
- if $z \ne z'$, then either $\gamma_z \cap \gamma_{z'} = \emptyset$, or $x(z) = x(z')$ and $\gamma_z \cap \gamma_{z'} = \{x(z)\} \subset \partial X$;
- each arc γ_z lies within an ε-ball centered at its basepoint $x(z) \in \partial X$: $\gamma_z \subset B_\varepsilon(x(z))$.

Observe that by the last property, N^ε is included in the ε-neighbourhood X_ε of X.

We identify $\mathbb{R}^2 = \mathbb{C}$ and $\mathbb{C} \subset \hat{\mathbb{C}}$ in the usual way and denote $U = \hat{\mathbb{C}} \setminus X$. Then U satisfies the hypotheses of Theorem B, so there exists a continuous map $\phi\colon \overline{\mathbb{D}} \to \overline{U}$ (the

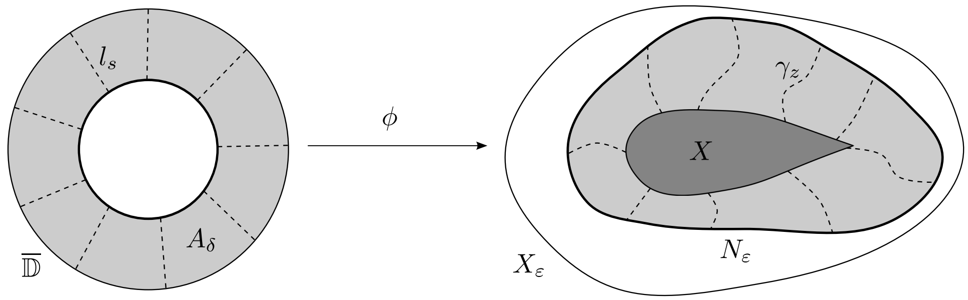


FIGURE 4. Proof of Theorem 1.1: The extended inverse Riemann map $\phi\colon \overline{\mathbb{D}} \to \hat{C} \setminus \mathrm{int}(X)$ maps the annulus $A_\delta = \{x \in \mathbb{C} \colon 1-\delta \le \|x\| \le 1\}$ onto $N^\varepsilon \setminus \mathrm{int}(X)$, such that $\partial\mathbb{D}$ is mapped onto ∂X. Each leaf γ_z of the foliation of $N^\varepsilon \setminus \mathrm{int}(X)$ is the image under ϕ of a radial line segment $l_s \subset A_\delta$.

extended inverse Riemann map), whose restriction to the open unit disk $\mathbb{D}$ is a conformal homeomorphism from $\mathbb{D}$ to U and $\phi(\partial\mathbb{D}) = \partial U$. The map ϕ can be chosen such that $\phi(0) = \infty$. Since $\overline{\mathbb{D}}$ is compact, ϕ is uniformly continuous, so we can find $\delta = \delta(\varepsilon) > 0$, such that for all $x, y \in \overline{\mathbb{D}}$ with $\|x-y\| < \delta$, $\|\phi(x) - \phi(y)\| < \varepsilon$.

Assume without loss of generality that $\delta < 1$ and set $A_\delta = \{x \in \mathbb{C} \colon 1-\delta \le \|x\| \le 1\}$ and $N^\varepsilon = \phi(A_\delta) \cup X$. Then N^ε is a closed neighbourhood of X, whose boundary $\Gamma = \partial N^\varepsilon = \phi(\{x \in \mathbb{D} \colon \|x\| = 1-\delta\})$ is a Jordan curve. We can then construct the foliation $\{\gamma_z\}_{z\in\Gamma}$ of $\overline{N^\varepsilon \setminus X}$ by taking the images of radial lines in A_δ (see Fig.4):

$$\gamma_z = \phi(\{r \cdot v \colon 1-\delta \le r \le 1\}), \quad \text{where } v = \frac{\phi^{-1}z}{\|\phi^{-1}z\|}.$$

All the required properties of N^ε and its foliation $\{\gamma_z\}$ then follow because ϕ maps $A_\delta \setminus \partial\mathbb{D}$ bijectively and uniformly continuously onto $N^\varepsilon \setminus X$; in particular the last required property follows by the choice of $\delta = \delta(\varepsilon)$.

Next, since f is a homeomorphism, $Y = f(X)$ is simply connected, compact and also locally connected (see [13, Theorem 17.15]). Hence, with $V = \hat{\mathbb{C}} \setminus Y$ and $\phi\colon \overline{\mathbb{D}} \to \overline{V}$ the corresponding extended inverse Riemann mapping, we can repeat the same construction as before to obtain a closed neighbourhood M^ε of Y, such that $\Theta = \partial M^\varepsilon$ is a Jordan curve and $\{\theta_z\}$ is a foliation of $\overline{M^\varepsilon \setminus X}$ with the same properties as $\{\gamma_z\}$.

By choosing $\delta > 0$ small enough, such that both constructions work with this given value, we get that $N^\varepsilon = X \cup \phi(A_\delta)$ and $M^\varepsilon = Y \cup \psi(A_\delta)$. Denote by $\{l_s \colon s \in S^1\}$ the foliation of A_δ by radial lines, such that $\gamma_z = \phi(l_s)$ for $z = \phi((1-\delta)\cdot s)$ and $\theta_{z'} = \psi(l_s)$ for $z' = \psi((1-\delta)\cdot s)$.

From Theorem C we know that f induces a homeomorphism $\hat{f}\colon S^1 \to S^1$, such that

$$\psi|_{S^1} \circ \hat{f} = f|_{\partial U} \circ \phi|_{S^1}.$$

Hence we can extend $f\colon X \to Y$ to $F_\varepsilon\colon N^\varepsilon \to M^\varepsilon$ by mapping each arc $\gamma_z \subset \overline{N^\varepsilon \setminus X}$ to the corresponding arc $\theta_{z'} \subset \overline{M^\varepsilon \setminus X}$. More precisely, let $H\colon A_\delta \to A_\delta$ be the homeomorphism which maps the radial line segment l_s linearly to the radial line segment $l_{\hat{f}(s)}$ (i.e., in polar coordinates $H = \mathrm{id}|_{[1-\delta,1]} \times \hat{f}$). Then define the homeomorphic extension of f to N^ε by setting

$$F_\varepsilon(x) = \begin{cases} f(x) & \text{if } x \in X, \\ \psi \circ H \circ \phi^{-1}(x) & \text{if } x \in N^\varepsilon \setminus X. \end{cases}$$

Note that for $0 < \varepsilon' < \varepsilon$, we can repeat the above construction to obtain Jordan domains $N^{\varepsilon'}$ and $M^{\varepsilon'}$ and a homeomorphism $F_{\varepsilon'}: N^{\varepsilon'} \to M^{\varepsilon'}$, and moreover $F_\varepsilon(x) = F_{\varepsilon'}(x)$ for all $x \in N^{\varepsilon'} \subset N^\varepsilon$.

Every connected component of $X \cap Y$ is a subset of a connected component of $N^\varepsilon \cap M^\varepsilon$. Let K^ε be the connected component of $N^\varepsilon \cap M^\varepsilon$ which contains C.

Claim 1. *$F_{\varepsilon'}$ has got a fixed point in $K^{\varepsilon'}$ for every $\varepsilon' \in (0, \varepsilon]$.*

This follows directly from the main result of the previous section: the periodic point for f in C is also a periodic point for $F_{\varepsilon'}$ in $K^{\varepsilon'}$. By Theorem 2.6, $F_{\varepsilon'}$ has a fixed point in $K^{\varepsilon'}$.

Claim 2. *F_ε has got a fixed point in $K^\varepsilon \cap (X \cap Y)$.*

By Claim 1 and the fact that for $0 < \varepsilon' < \varepsilon$, $F_{\varepsilon'}$ is the restriction of F_ε, F_ε has got a sequence of fixed points $(p_k)_{k\in\mathbb{N}}$, such that $p_k \in N^{\varepsilon/k} \cap M^{\varepsilon/k}$. By passing to a convergent subsequence, we get that $p_k \to p$ as $k \to \infty$, with $p \in K^\varepsilon \cap (X \cap Y)$ and $F_\varepsilon(p_k) = F_{\varepsilon/k}(p_k) = p_k$. By continuity $F_\varepsilon(p) = p$, which proves the claim.

Claim 3. *F_ε has got a fixed point in C.*

By Claim 2 we know that F_ε (and hence $F_{\varepsilon'}$ for $\varepsilon' \in (0, \varepsilon]$) possesses a fixed point $p \in K^\varepsilon \cap (X \cap Y)$. If $p \notin C$, then p lies in another connected component of $K^\varepsilon \cap (X \cap Y)$, say C'. We can now apply Lemma 3.1 to $K^\varepsilon \cap (X \cap Y)$ and its connected components. Since $N^{\varepsilon'} \subseteq X_{\varepsilon'}$ and $M^{\varepsilon'} \subseteq Y_{\varepsilon'}$, we obtain that for $\varepsilon' > 0$ sufficiently small, $p \notin K^{\varepsilon'}$. By again applying Theorem 2.6 and Claim 2, we get that $F_{\varepsilon'}$ has a fixed point $p' \in K^{\varepsilon'} \cap (X \cap Y)$, which is also a fixed point for F_ε.

We can iterate the argument and obtain a sequence of F_ε-fixed points $(p_k)_{k\in\mathbb{N}}$, such that $p_k \in K^{1/k} \cap (X \cap Y)$. As in the proof of the previous claim, by passing to a convergent subsequence, $p_k \to p$ as $k \to \infty$, we get an F_ε-fixed point $p \in K^\varepsilon \cap (X \cap Y)$. If p lies in a connected component of $K^\varepsilon \cap (X \cap Y)$ distinct from C, then, by Lemma 3.1, $p \notin K^{\varepsilon'}$ for sufficiently small ε'. But by construction, $p \in K^{\varepsilon'}$ for all $\varepsilon' \in (0, \varepsilon]$, so $p \in C$, finishing the proof of Claim 3.

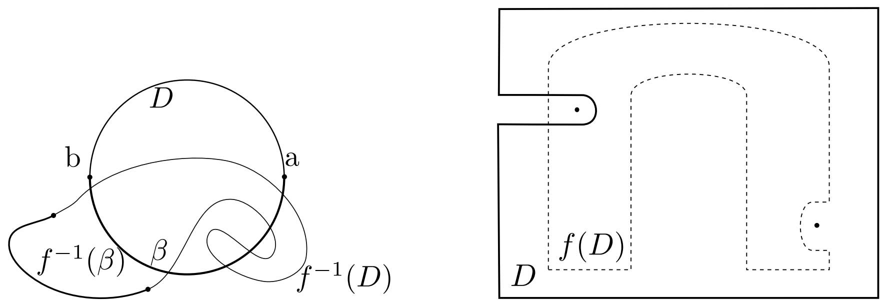
Claim 3 implies that f has a fixed point in C, and Theorem 1.1 is proved. □

Proof of Corollary 1.2. If C is a connected component of $X \cap Y$ which contains a point x such that $f^n(x) \in C$ for all $n \in \mathbb{N}$, then exactly as in the proof of Theorem 1.1, we can construct an extension F of f to a closed neighbourhood N^ε of X, for $\varepsilon > 0$ sufficiently small. Using Corollary 2.7 instead of Theorem 2.6, we get that the extension has got a fixed point. The same argument then shows that one such fixed point lies in C, finishing the proof of the corollary. □

4. Discussion of Assumptions and Possible Extensions

Easy examples show that in our main results we cannot omit the hypothesis of a periodic orbit being completely contained in a given connected component of $X \cap f(X)$: take X to be a closed ε-neighbourhood of the straight line segment $S = [-1, 1] \times \{0\}$ and let $f: X \to Y$ be a homeomorphism which maps S to the semicircle $\{(x, y): x^2 + y^2 = 1, y \geq 0\}$ with $f(-1, 0) = (1, 0)$ and $f(1, 0) = (-1, 0)$. For $\varepsilon > 0$ small, f has got a period-two orbit (spread over two different connected components of $X \cap f(X)$) but no fixed point.

Furthermore, obvious counterexamples show that our main result, Corollary 1.2, fails when the domain $X \subset \mathbb{R}^2$ of the homeomorphism f is not compact. Also, we are not aware of generalizations to higher dimensions; in fact, simple counterexamples to Theorem 1.1 in its current form can be constructed, when $X \subset \mathbb{R}^n$ is an n-dimensional ball with $n \geq 3$.



(A) (B)

FIGURE 5. (A) Jordan domain satisfying Bonino's [2] condition.
(B) Horseshoe map with neighbourhood of a period-two point removed from its domain: The map has got period-three points of non-trivial braid type, but no period-two points, showing that a simple generalization of period-forcing results to non-self maps of planar domains does not work.

Bonino [2] showed that if an orientation-preserving homeomorphism $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ possesses non-escaping points in a topological disk D bounded by a simple closed curve C, and C can be split into two arcs $\alpha = [a, b]_C$, $\beta = [b, a]_C$ such that $D \cap f^{-1}(\beta) = \emptyset$ and $f^{-1}(D) \cap \alpha = \emptyset$ (Fig.5(A)), then f has got a fixed point *in D*. Bonino's theorem is similarly topological, but does not imply (and is not implied by) our results.

Another result related to Theorem 1.1 is due to Brown [6, Theorem 5]. Brown assumed that for the n-periodic point $x \in X$ there is a connected neighbourhood W of $\{x\} \cup \{f(x)\}$ with $f^i(W) \subset X$ for $i = 1, \dots, n$, and showed that f then has a fixed point in X.

A somewhat different class of results relates to the question of period forcing. Gambaudo et al [9] showed that a C^1 orientation-preserving embedding of the disk into itself has periodic orbits of all periods, if it has a period-three orbit of braid type different than the rotation by angle $2\pi/3$ (see also Kolev [11] for a topological version of this result). For $f(X) \not\subset X$, this statement is false: a counterexample is a version of the Smale horseshoe map, where a period-two point is removed from the disc together with a narrow strip such that the remaining domain X is still simply connected (see Fig.5(B)). Then f, the restriction of the horseshoe map to X, is an orientation-preserving homeomorphism with $X \cap f(X)$ connected, and one can choose the removed strip narrow enough such that f still has got period-three orbits but no period-two orbit in X. We conclude with the following question:

Question. *Can one find conditions on X and $f(X)$, under which **every extension** of f to a homeomorphism of $\mathbb{R}^2$ has periodic orbits of every period **passing through** X, whenever f has got a period-three point in X (of braid type different than the rotation by angle $2\pi/3$)?*

The method of extending a non-self map of a compact set to a self-map of $\mathbb{R}^2$ or a disk without additional fixed (or periodic) points, could possibly also be applied to this problem.

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Appendix. Proofs of Technical Lemmas

Proof of Lemma 2.4. We apply Proposition 2.3 with $U = D$ and $U' = E$. The set $D \cap E$ is a Jordan domain and by (1), its boundary curve consists of the (pairwise disjoint) sets

- $\partial D \cap \partial E$ (isolated points or closed arcs),
- $\alpha_i \subset \partial D$, $i \in I$ (open arcs), and
- $\beta_j \subset \partial E$, $j \in J$ (open arcs).

It also follows from Proposition 2.3, that $\overline{E \setminus D}$ is the union of Jordan domains A_i bounded by $\alpha_i \cup [a_i, b_i]_{\partial E}$ and $\overline{D \setminus E}$ is the union of Jordan domains B_j bounded by $\beta_j \cup [c_j, d_j]_{\partial D}$ (see Fig.1).

By Schoenflies' theorem, each Jordan domain A_i is homeomorphic to the closed unit disk $\overline{\mathbb{D}} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$. Let $h_i : A_i \to \overline{\mathbb{D}}$ be such homeomorphism. We can assume $h_i(a_i) = (-1, 0)$ and $h_i(b_i) = (1, 0)$. Then the partition of $\overline{\mathbb{D}}$ into vertical line segments $l_s := \{(x, y) \in \overline{\mathbb{D}} : x = s\}$, $s \in (-1, 1)$, gives rise to a partition $\{h_i^{-1}(l_s) : s \in (-1, 1)\}$ of A_i into closed arcs, each connecting a point of α_i to a point of $(a_i, b_i)_{\partial E}$. Similarly one can obtain a partition of B_j into closed arcs connecting points on β_j to points on $(c_j, d_j)_{\partial D}$.

Note that $D \cup E$ is also a Jordan domain and that the $(a_i, b_i)_{\partial E}$, $i \in I$, and $(c_j, d_j)_{\partial D}$, $j \in J$, together with $\partial D \cap \partial E$ form a partition of its boundary. Thus we obtained a collection of (pairwise disjoint) closed arcs, each connecting precisely one point of $\partial(D \cap E)$ to precisely one point of $\partial(D \cup E)$. Denote the arc corresponding to $z \in \partial(D \cup E)$ by l^z, and let $l^z = \{z\}$ whenever $z \in \partial D \cap \partial E$.

Since $D \cup E$ is a Jordan domain, we can again apply Schoenflies' theorem to obtain a homeomorphism $h : \mathbb{R}^2 \setminus \operatorname{int}(D \cup E) \to \mathbb{R}^2 \setminus \mathbb{D}$. Let r_θ be the radial line segment in $\mathbb{R}^2 \setminus \mathbb{D}$ expressed in polar coordinates as $\{(r, \phi) : r \geq 1, \phi = \theta\}$. Then $\{r_\theta : \theta \in [0, 2\pi)\}$ forms a partition of $\mathbb{R}^2 \setminus \mathbb{D}$, which can be pulled back to a partition $\{h^{-1}(r_\theta) : \theta \in [0, 2\pi)\}$ of $\mathbb{R}^2 \setminus \operatorname{int}(D \cup E)$. This partition consists of (pairwise disjoint) arcs m^z, each connecting a point on $z \in \partial(D \cup E)$ to ∞.

Combining the above, we get that each point $z \in \partial(D \cup E)$ is the endpoint of two uniquely defined arcs l^z and m^z (l^z possibly being the trivial arc $\{z\}$). Let $L^z := l^z \cup m^z$, which is an arc connecting a point on $\partial(D \cap E)$ to ∞, for every $z \in \partial(D \cup E)$. Then $\{L^z : z \in \partial(D \cup E)\}$ is a partition of $\mathbb{R}^2 \setminus \operatorname{int}(D \cap E)$ with the desired properties. □

Proof of Lemma 3.1. We denote the minimal distance between any two points of two disjoint sets $U, V \subset \mathbb{R}^n$ by

$$\mathrm{d}(U, V) = \inf_{x \in U,\ y \in V} \|x - y\|.$$

Suppose for a contradiction that for all $\varepsilon > 0$, C and C' lie in the same connected component of X_ε. Then for each fixed $\varepsilon > 0$, there exists $N(\varepsilon) \in \mathbb{N}$ and a finite chain of connected components of X

$$C = C_0^\varepsilon, C_1^\varepsilon, \ldots, C_{N(\varepsilon)}^\varepsilon = C', \quad \text{where } C_i^\varepsilon \in \mathcal{C} \text{ for } i = 0, 1, \ldots, N(\varepsilon),$$

such that

$$\mathrm{d}(C_i^\varepsilon, C_{i+1}^\varepsilon) < 2\varepsilon \text{ for } i = 0, 1, \ldots, N(\varepsilon) - 1. \tag{4.1}$$

(Note that (4.1) implies that $(C_i^\varepsilon)_\varepsilon \cap (C_{i+1}^\varepsilon)_\varepsilon \neq \emptyset$.)

Now for $k \in \mathbb{N}$ let $\varepsilon_k = 1/k$ and define

$$S_k := \bigcup_{i=0}^{N(\varepsilon_k)} C_i^{\varepsilon_k}.$$

We use the fact that the space of all non-empty compact subsets of a compact complete metric space, equipped with the Hausdorff distance, forms itself a compact complete metric space. Since $(S_k)_{k\in\mathbb{N}}$ is a sequence of non-empty compact subsets of $X \subset \mathbb{R}^n$, it has a convergent subsequence and by passing to this subsequence we can assume $S_k \to S$ as $k \to \infty$, for some non-empty compact set $S \subseteq X$.

Claim. *$C \subset S$, $C' \subset S$, and S is connected.*

The inclusions follow immediately from the fact that $C \subset S_k$ and $C' \subset S_k$ for all $k \in \mathbb{N}$. Further, suppose S is disconnected. Then S can be written as $S_1 \cup S_2$, where S_1 and S_2 are non-empty disjoint open sets in X, i.e., there exist disjoint open sets $U_1, U_2 \subset \mathbb{R}^n$, such that $S_1 = U_1 \cap X$ and $S_2 = U_2 \cap X$. Note that by compactness of S, the sets U_1 and U_2 can be chosen such that $\overline{U_1} \cap \overline{U_2} = \emptyset$, hence

$$\mathrm{d}(U_1, U_2) =: d > 0. \tag{4.2}$$

Since S is contained in the open set $U_1 \cup U_2$, $S_k \to S$ implies that $S_k \subset (U_1 \cup U_2)$ for $k \in \mathbb{N}$ sufficiently large. But by (4.2) the set $U_1 \cup U_2$ has a 'gap' of size $\geq d$, whereas the set S_k admits no such 'gaps' wider than $2/k$ by (4.1). For $k \geq 2/d$ this is a contradiction, and the claim is proved.

We get that S is a connected subset of X containing both C and C', which contradicts the assumption that C and C' are distinct connected components of X. This finishes the proof of the lemma. □

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